MATHEMATICS

COMPLETENESS OF $C_{\mathscr{S}}(X)$

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INTRODUCTION

In this note we discuss the completeness of the function space $C_{\mathscr{S}}(X)$, motivated by a problem of S. T. M. ACKERMANS (See (1)). $C_{\mathscr{S}}(X)$ will have the uniformity of uniform convergence, $\mathbf{R}$ having its usual uniformity.

1. NOTATION AND DEFINITIONS

Let f be a real valued function on X. The non-zero set of f is the set $N(f)=\{x|f(x)\neq 0\}$, it is also called the cozero set of f. The support of f is the set $S(f)=\mathrm{cl}_X[N(f)]$. Any set which is the support of some f will be called a support set. We say that N is the cozero set of a support set S if $N=N(f)$ and $S=S(f)$ for some f.

We shall denote by $\mathscr{S}$ a family of closed subsets of a topological space satisfying

$\mathscr{S}_1$. $S\subset S_1\in\mathscr{S}$ and S is closed implies $S\in\mathscr{S}$.

$\mathscr{S}_2$. $S_1, S_2\in\mathscr{S}$ implies $S_1\cup S_2\in\mathscr{S}$.

$\mathscr{S}$ is a normal family if it also satisfies

$\mathscr{S}_3$. $S_1, S_2\in\mathscr{S}$ and $S_1\subset \mathrm{int}\,(S_2)$ implies $S_1\subset\mathrm{int}\,(S_3)$ and $S_3\subset\mathrm{int}\,(S_2)$ for some $S_3\in\mathscr{S}$.

We denote by $C_{\mathscr{S}}(X)$ the family of continuous real valued functions on X with support in $\mathscr{S}$. $C_{\mathscr{S}}(X)$ will have the uniformity of uniform convergence, so that a basis for the uniformity consists of the sets $\{(f, g)|\sup_x |f(x)-g(x)|<r\}$ f, g with support in $\mathscr{S}$, $r>0$.

If $\mathscr{G}$ is a family of subsets of X, a G_σ set is a countable union of members of $\mathscr{G}$. The family of regularly closed members of $\mathscr{G}$ will be denoted by $\mathscr{R}\mathscr{G}$, a set being regularly closed if it is the closure of its interior.

2. COMPLETENESS OF $C_{\mathscr{S}}(X)$

To decide on completeness of $C_{\mathscr{S}}(X)$ it is sufficient to consider only Cauchy sequences, since every Cauchy net in $C_{\mathscr{S}}(X)$ has a subnet which is a Cauchy sequence.

2.1 Proposition. $C_{\mathscr{S}}(X)$ is complete in the uniformity of uniform convergence iff every countable union of support sets in $\mathscr{S}$ has its closure in $\mathscr{S}$.

Proof. We first prove sufficiency. Let $\{f_n\}$ be a Cauchy sequence in $C_{\mathscr{S}}(X)$. f_n converges to a continuous real valued function f and $N(f) \subset \subset \bigcup N(f_n) \subset \bigcup S(f_n)$. Assuming $\mathrm{cl}\,(\bigcup S(f_n)) \in \mathscr{S}$, we have $S(f) = \mathrm{cl}\,(N(f)) \in \mathscr{S}$. Hence $C_{\mathscr{S}}(X)$ is complete. Conversely, let $\{S(f_n)\}$ be a countable family of support sets in $\mathscr{S}$. $s_n = \sum_1^n 2^{-p}(|f_p| \wedge 1) \in C_{\mathscr{S}}(X)$ since $S(f_n) = S(|f_n| \wedge 1)$ and $S(s_n) = \bigcup_1^n S(f_p) \in \mathscr{S}$. $\{s_n\}$ is a Cauchy sequence in $C_{\mathscr{S}}(X)$, hence $s_n \to s = \sum_1^\infty 2^{-n}(|f_n| \wedge 1) \in C_{\mathscr{S}}(X)$. We have $\bigcup N(f_n) = N(s)$ and $\mathrm{cl}\,(\bigcup S(f_n)) = \mathrm{cl}\,(\bigcup N(f_n))$, hence $\bigcup S(f_n) \subset S(s) \in \mathscr{S}$. This completes the proof.

In the following propositions we characterise the support sets in $\mathscr{S}$.

2.2 Proposition. Every support set S in $\mathscr{S}$ is a regularly closed set such that $\mathrm{int}\,(S)$ includes a dense RS_σ set.

Proof. Let $S = S(f)$ be in $\mathscr{S}$. $N(f) = \bigcup C_n$, where $C_n = \mathrm{cl}\,\{x|f(x)| > n^{-1}\}$ is a regularly closed member of $\mathscr{S}$, and $S(f) \supset \mathrm{int}\,(S(f)) \supset N(f)$. The proof is complete.

If $\mathscr{S}$ is a normal family the converse of proposition 2.2 holds. To prove this we need the following

2.3 Proposition. Let $\mathscr{S}$ be a normal family and S_1, S_2 in $\mathscr{S}$ be such that $S_1 \subset \mathrm{int}\,(S_2)$. There is a continuous function $f\colon X \to [0, 1]$ such that $S_1 \subset f^{-1}[1]$ and $X - \mathrm{int}\,(S_2) \subset f^{-1}[0]$.

Proof. As in Urysohn's lemma, one constructs a continuous function $g\colon S_2 \to [0, 1]$ which is 1 on S_1 and 0 on $S_2 - \mathrm{int}\,(S_2)$. Thus g is zero on the boundary of S_2 and, hence, the function f defined by $f|S_2 = g$ and $f|X - S_2 = 0$ is a continuous function on X. f satisfies the requirements in the proposition.

2.4 Proposition. Let $\mathscr{S}$ be a normal family. S is a support set in $\mathscr{S}$ if S is regularly closed and $\mathrm{int}\,(S)$ contains a dense S_σ set.

Proof. Let S be regularly closed and let $\{C_n\}$ be a family of sets in $\mathscr{S}$ such that $\bigcup C_n$ is dense in $\mathrm{int}\,(S)$. Now $C_n \subset \mathrm{int}\,(S)$ and S is in $\mathscr{S}$, so there is a continuous function $f_n\colon X \to [0, 1]$ such that $C_n \subset N(f_n) \subset \subset S(f_n) \subset \mathrm{int}\,(S)$. The function $s_n = \sum_1^n 2^{-r} f_r$ is continuous and converges uniformly to $s = \sum_1^\infty 2^{-n} f_n$, hence s is continuous. Now $\bigcup C_n \subset \bigcup N(f_n) = N(s)$ and $N(s) \subset \mathrm{int}\,(S)$, it follows that $S = S(s)$.

We note that when $\mathscr{S}$ is a normal family and S is in $\mathscr{S}$, the condition that $\mathrm{int}\,(S)$ contain a dense S_σ is equivalent to the apparently stronger one that it contain a dense RS_σ.

When $\mathscr{S}$ is a normal family there is a characterisation of X for which $C_{\mathscr{S}}(X)$ is complete in terms of $\mathscr{S}$ alone. This is an immediate consequence of propositions 2.1, 2.2 and 2.4.

2.5 Example. We note that a support set in $\mathscr{S}$ need not have an interior which is an S_σ set even when $\mathscr{S}$ is a normal family consisting of compact sets. Let $\mathscr{S}$ be the family of compact subsets of X, the space obtained by identifying $(\Omega, 0) \in \Omega' \times I$ and $(\Omega, 0) \in \{\Omega\} \times I$ in $(\Omega' \times I) \vee \vee (\{\Omega\} \times I)$. Here Ω is the first uncountable ordinal, Ω' its successor, with the order topology; $\vee$ denotes disjoint union; I is $[0, 1]$ with the usual topology. Define $f: X \to I$ by $f \circ i(\sigma, t) = t$ for $(\sigma, t) \in \Omega' \times I$, and $f \circ i(\Omega, t) = 0$ for $(\Omega, t) \in \{\Omega\} \times I$; where i is the quotient map onto X. Now $S(f)$ is in $\mathscr{S}$, and $\operatorname{int}(S(f)) = i[\{(\sigma, t) | \sigma < \Omega\} \cup \{(\Omega, t) | t \neq 0\}]$; this set is not a countable union of compact sets.

2.6 Proposition. Let $U = \bigcup \{\operatorname{int}(S) | S \in \mathscr{S}\}$. If every member of $\mathscr{S}$ is contained in a countable union of support sets in $\mathscr{S}$, then the following conditions are equivalent to completeness of $C_{\mathscr{S}}(X)$.
1. Any S_σ subset of U is contained in a member of $\mathscr{S}$.
2. Any S_σ subset of U is contained in a member of $\mathscr{R}\mathscr{S}$.

Proof. We shall prove that $1 \Rightarrow$ completeness $\to 2$. Assume 1. Let $\{S_n\}$ be a countable family of support sets in $\mathscr{S}$. By 2.2, $\operatorname{int}(S_n)$ contains a dense S_σ set. Hence $\bigcup \operatorname{int}(S_n)$ contains a dense S_σ set; and this set is a subset of U since $\operatorname{int}(S_n) \subset U$. By 1, there is a member of $\mathscr{S}$ which contains $\operatorname{cl}(\bigcup \operatorname{int}(S_n)) = \operatorname{cl}(\bigcup S_n)$. It follows from 2.1 that $C_{\mathscr{S}}(X)$ is complete. Conversely, assume $C_{\mathscr{S}}(X)$ is complete and let $\{S_n\}$ be a countable family of members of $\mathscr{S}$. From our assumptions on $\mathscr{S}$, $\bigcup S_n \subset \bigcup C_m$, $\{C_m\}$ being a countable family of support sets in $\mathscr{S}$. Hence $\bigcup C_m$ is contained in a member of $\mathscr{S}$, by 2.1. Since each C_m is regularly closed, it follows that $\bigcup C_m$ has a regularly closed closure and 2. is proved.

3. Examples

We now give two examples and state the corresponding particular cases of proposition 2.6. In both examples $X = U = \{\operatorname{int}(S) | S \in \mathscr{S}\}$.

3.1 Let X be a regular topological space which is locally Lindelöf. Using the fact that a regular Lindelöf space is normal, it follows that the family $\mathscr{L}$, of closed Lindelöf subsets, is a normal family. We shall prove that any member of $\mathscr{L}$ is included in a countable union of support sets in $\mathscr{L}$.

Consider L in $\mathscr{L}$. A point of L has a regularly closed Lindelöf neighbourhood V. V is regular and Lindelöf, hence normal, so there is a continuous $f: V \to [0, 1]$ such that f is 1 at the point and 0 on the boundary of V. Hence f has a continuous extension $g: X \to [0, 1]$, with $S(f) = S(g)$; and $S(g)$ is in $\mathscr{L}$, being a closed subset of V. The interiors of these sets cover L, so there is a countable family of support sets covering L.

Hence the following proposition.

3.1 Proposition. Let X be a regular and locally Lindelöf space. $C_{\mathscr{L}}(X)$ is complete iff the union of every countable family of closed Lindelöf subsets has a Lindelöf closure.

3.2 Let X be locally compact and regular, and let $\mathscr{C}$ be the family of closed and compact subsets. We have

3.2 Proposition. (See (2), problem 7 G.) Let X be locally compact and regular. $C_{\mathscr{C}}(X)$ is complete iff any countable union of compact sets has compact closure.

3.3 Proposition. Let X_1 and X_2 be non empty, regular, locally compact spaces. $C_{\mathscr{C}}(X_1 \times X_2)$ is complete iff $C_{\mathscr{C}}(X_1)$ and $C_{\mathscr{C}}(X_2)$ are complete.

Proof. Assume $C_{\mathscr{C}}(X_i)$ is complete, $i=1,2$. Let $\{C_n\}$ be a countable family of compact subsets of $X_1 \times X_2$, which is a regular and locally compact space. Then the sets of projections $\{p_1(C_n)\}$ and $\{p_2(C_n)\}$ are countable families of compact subsets of X_1 and X_2, respectively. Let K_i be a compact subset of X_i such that $\bigcup p_i(C_n) \subset K_i$, $i=1,2$. Then $p_i(\bigcup C_n) \subset K_i$, hence $\bigcup C_n \subset K_1 \times K_2$. From 3.2 it follows that $C_{\mathscr{C}}(X_1 \times X_2)$ is complete since $K_1 \times K_2$ is compact.

Conversely, assume $C_{\mathscr{C}}(X_1 \times X_2)$ complete and let $\{C_n\}$ be a countable family of closed compact subsets of X_1. Let b be a point in X_2. Then $\{C_n \times \{b\}\}$ is a countable family of compact subsets of $X_1 \times X_2$, hence there is a compact set K such that $\bigcup (C_n \times \{b\}) \subset K$. Now $p_1(\bigcup C_n \times \{b\}) = \bigcup p_1(C_n \times \{b\}) = \bigcup C_n \subset p_1(K)$, and $p_1(K)$ is compact. Hence $C_{\mathscr{C}}(X_1)$ is complete. Similarly, $C_{\mathscr{C}}(X_2)$ is complete.

From 3.2 it follows that a regular locally compact space for which $C_{\mathscr{C}}(X)$ is complete is countably compact (hence pseudocompact). The converse is not true: let $X = \beta\mathbf{N} - \{p\}$, where $\beta\mathbf{N}$ is the Stone-Čech compactification of the discrete set of integers and $p \in \beta\mathbf{N} - \mathbf{N}$. X is a locally compact and countably compact Hausdorff space, but $C_{\mathscr{C}}(X)$ is not complete since $\mathbf{N}$ is σ-compact and $\mathrm{cl}\,(\mathbf{N}) = X$ is not compact, so that proposition 3.2 applies.

Every topologically complete and countably compact space is compact. In particular, so is every paracompact, or every metric, countably compact space (See (4), Ch. 6, L. and M.). Hence the following propositions.

3.4 Proposition. (See (3) (11.43) (e)). If X is a locally compact topological group, then X is compact iff $C_{\mathscr{C}}(X)$ is complete.

Proof. A locally compact topological group is paracompact (See (3) (8.13), or (4), Ch. 5, Y.).

3.5 Proposition. (See (1)). If X is a locally compact metric space, X is compact iff $C_{\mathscr{C}}(X)$ is complete.

However, Ω, the first uncountable ordinal with its order topology, is a locally compact, non-compact normal space and $C_{\mathscr{C}}(\Omega)$ is complete.

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